

# Strings, paths, and standard tableaux

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## Abstract

For the vacuum sectors of regime-III ABF models, we observe that two sets of combinatorial objects: the strings which parametrize the row-to-row transfer matrix eigenvectors, and the paths which parametrize the corner transfer matrix eigenvectors, can both be expressed in terms of the same set of standard tableaux. Furthermore, the momenta of the strings, the energies of the paths, and the charges of the tableaux are such that there is a weight-preserving bijection between the two sets of eigenvectors, wherein the tableaux play an interpolating role. This bijection is so natural, that we conjecture that it exists in general.

## 1 Introduction

In statistical mechanics, physical quantities are computed by averaging dynamical variables over an ensemble of all possible state configurations. Each configuration is weighted by a statistical (Boltzmann) factor. The transfer matrix is a convenient tool to generate the set of all properly-weighted configurations. In exactly-solvable statistical models on two-dimensional lattices, the corner, and row-to-row transfer matrices (CTM and RTM, respectively) are particularly useful.<sup>1</sup>

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<sup>1</sup>For an introduction to exactly solvable lattice models and transfer matrices, see [1].

The purpose of this work is to point out a relation between the set of *paths* which parametrize the eigenvectors of the CTM [1, 2, 3], and the set of *strings*, which play the same role for the RTM. [4]. The fact that such a relation must exist is not new. In fact, in our point of view, it lurks in the background of all recent works on *boson-fermion* character identities.<sup>2</sup> What is new in this work is the establishment of an explicit weight-preserving bijection between the relevant combinatorial objects: The two sets of eigenvectors, and their eigenvalues, in which, to our delight, the ubiquitous standard tableaux play a central role. In the following, we briefly outline the contents and logic of the paper.

We will restrict our attention to the anti-ferromagnetic (regime III) ABF models [2], because they are simple, well-understood and for our purposes, representative of the general situation. These models form an infinite series, labelled by an integer  $\ell = 1, 2, \dots$ . The  $\ell = 2$  model is the Ising model (the  $\ell = 1$  model being trivial). The  $\ell \geq 3$  models are *multi-critical*  $Z_2$  models.<sup>3</sup> Working in a certain model, one can specify *boundary conditions*, both on the outer boundary of the lattice, and the value of a single variable at the center of the lattice. Each choice of boundary conditions corresponds to a choice of *sector* in the space of allowed lattice configurations. For the sake of simplicity, in this work, we restrict our attention to the *vacuum sectors* of the ABF models.<sup>4</sup>

The CTM eigenvectors can be parametrized in terms of one-dimensional configurations called *paths*, that will be defined in §2.1. The corresponding eigenvalues, which turn out to be non-negative integers, are called *energies*, and can be used as weights on the paths. The weighted sum over all paths that belong to a certain sector in a certain model turns out to be the character of a highest weight module (HWM) of an infinite dimensional algebra. In the case of the ABF models, they are HWM's of Virasoro algebras. This sum can be evaluated by solving the recurrence relations satisfied by the generating function of the weighted paths. The resulting expressions are in the form of alternating-sign q-series. Such alternating-sign expressions are referred to in the physics literature as *bosonic*.<sup>5</sup>

In §2, we relate the first set of combinatorial objects that we are interested in: the paths, to standard tableaux, and define weights on the tableaux, or tableaux statistics, that are directly related to the weights on the paths. The reason we aimed at relating the paths to the tableaux is that the second class of combinatorial objects that we are interested in: the strings or rigged configurations<sup>6</sup>, which label the RTM eigenvectors, and with which we wish to associate the paths, are also known to be related to standard tableaux [9].

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<sup>2</sup>For a survey of the early literature on this topic, see [5]. For a more recent, and relatively complete compendium of references, see [6, 7].

<sup>3</sup>For an explanation of what these words mean, and further details regarding the physical interpretation of these models, we refer to [8].

<sup>4</sup>What we call the vacuum sector corresponds to the choice  $r = s = 1$ , of the parameters of [2].

<sup>5</sup>For a discussion of the origin of this terminology, we refer to [5].

<sup>6</sup>We shall be using the terms strings and rigged configurations interchangeably in this paper.

In §3, we recall the definition of rigged configurations: partitions that are equipped, or *rigged*, with a set of integers. Both the partitions, and the corresponding rigging integers satisfy certain model-dependent, and sector-dependent conditions. Next, we outline a celebrated bijection between the rigged configurations and standard tableaux [9], and recall the relation between the rigging integers, and the tableaux statistics mentioned above [10].

Summing over the rigged configurations, of a certain sector, in a certain model, where each configuration is weighted by a sum involving the rigging integers, one obtains yet another expression for the character of the corresponding HWM. Such expressions differ from those computed using CTM paths in that they turn out to be constant-sign expressions. These expressions are often referred to, in the physics literature, as *quasi-particle*, or *fermionic* expressions [11, 12].<sup>7</sup>

Now that we can express both sets of eigenvectors, and corresponding eigenvalues, in terms of tableaux and tableaux statistics respectively, we identify the two sets of tableaux in §4. This amounts to a bijective proof of the polynomial analogue of a boson-fermion character identity for each of the ABF models under consideration. The reason that we obtain polynomial, rather than q-series identities, is that we can consider paths, and rigged configurations, with an upper bound  $L$  on the size of the system. In the limit  $L \rightarrow \infty$ , we obtain yet another proof of the generalized Roger-Ramanujan identities conjectured in [13], and proven in [14, 15, 16].

Though we work in the limited context of the vacuum sector of regime III of the ABF models, the bijections that we establish are so simple that we conjecture that they extend to the entire spectrum. Furthermore, we believe that analogous bijections exist in more general models. These conjectures are supported by the larger number of conjectured, and in many cases proven, Rogers-Ramanujan-type identities, that are related to more general ABF sectors, and other models.

## 2 CTM objects: paths and tableaux

### 2.1 Paths

The weighted sum over all configurations of a certain lattice model, with fixed dynamical variables on the boundaries, and at the center of the lattice, defines a *one-point function*. The one-point functions of the ABF models can be evaluated using Baxter's CTM method.

Roughly speaking, the CTM method reduces the physical problem of evaluating a one-point function of a lattice model, to the mathematical problem of evaluating the generating function of a set of *paths* on a segment of the weight

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<sup>7</sup>For further details, also regarding the derivation of the strings, their physical interpretation, and in particular, the physical significance of the rigging integers, which are related to the *Takahashi numbers*, or quasi-particle momenta in the physics literature, we refer to [5].

lattice of the Lie algebra  $sl(2)$ . These paths are weighted in a certain way, and obey certain restrictions, that need not concern us here. All we need to know, is that the one-point functions depend on a certain *temperature-like* parameter, and that the eigenvectors of the CTM consist of linear combinations of paths [17]. These linear combinations can be labeled in terms of a single unique path [18]. In the zero temperature limit, each linear combination reduces to the single path that labels it. The weight of a path is the corresponding eigenvalue of the CTM in the zero temperature limit.

In the case of vertex models based on affine algebras, such as the six-vertex model, and the related restricted solid-on-solid models, such as the ABF models, these one-point functions can be expressed in terms of the characters of highest weight modules of affine and Virasoro algebras, respectively [3]. The CTM acts on the paths as the grading operator, or derivation, of the corresponding algebra [17].

The ABF model characterized by the integer  $\ell$  is defined in terms of *height variables* that live on the sites of a square lattice, and take values in the set of level  $\ell$  dominant integral weights of the algebra  $U_q \widehat{sl}_2$ . A path is a sequence of heights where successive heights differ by  $\pm 1$ . Equivalently, for the fundamental weights  $\Lambda_0, \Lambda_1$ , we introduce the set  $\mathcal{P}_\ell$  which is the set of level  $\ell$  weights:

$$\mathcal{P}_\ell := \{\Lambda = k\Lambda_0 + (\ell - k)\Lambda_1 | 0 \leq k \leq \ell\},$$

and define  $\mathcal{A} := \{\hat{0}, \hat{1}\}$  where  $\hat{0} := (\Lambda_1 - \Lambda_0)$  and  $\hat{1} := (\Lambda_0 - \Lambda_1)$ . Finally, introduce the automorphism  $\sigma$  of the weight lattice which acts on the fundamental weights by the  $\mathbf{Z}_2$  transformation:  $\Lambda_i \mapsto \sigma(\Lambda_i) = \Lambda_{1-i}$ .

**Definition 1 (path)** A  $\Lambda$ -path is a sequence  $p = (\mu_0, \mu_1, \dots, \mu_L)$ ,  $\mu_j \in \mathcal{P}_\ell$  such that  $\mu_{j+1} - \mu_j \in \mathcal{A}$ , and for  $j > L$ ,  $\mu_j = \mu_L + \sigma^{(j-L)}(\Lambda)$ . We denote the set of all such paths of length  $L$  as  $\mathbf{P}_\ell^L$ .

We can encode a path in terms of a sequence of zeroes and ones as follows:

**Definition 2 (0 - 1 sequences)** For a path  $p = (\mu_0, \dots, \mu_L) \in \mathbf{P}_\ell^L$  we define a sequence  $\eta(p) = (\eta_0, \dots, \eta_L)$ , where  $\eta_j = \mu_{j+1} - \mu_j$ .

In each sector, there is a unique path,  $\bar{p}$  for which the energy is lower than that of any other path in the same sector: *the ground state path*. In regime-III of the ABF models, which is what we are interested in,  $\eta(\bar{p})$  is given by  $\eta_j \equiv (\widehat{j+1}) \pmod{2}$ .

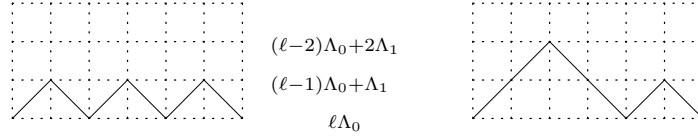
**Example 1 (ground state path)**  $\bar{p} \in \mathbf{P}_\ell$  (note, we do not specify  $L$  here),

$$\begin{aligned} \bar{p} &= (\ell\Lambda_0, (\ell-1)\Lambda_0 + \Lambda_1, \ell\Lambda_0, (\ell-1)\Lambda_0 + \Lambda_1, \ell\Lambda_0, (\ell-1)\Lambda_0 + \Lambda_1, \ell\Lambda_0, \dots) \\ \eta(\bar{p}) &= (\hat{0}, \hat{1}, \hat{0}, \hat{1}, \hat{0}, \hat{1}, \dots) \end{aligned}$$

**Example 2** A path  $p^{(1)} \in \mathbf{P}_\ell^L$ .

$$\begin{aligned} p^{(1)} &= (\ell\Lambda_0, (\ell-1)\Lambda_0 + \Lambda_1, (\ell-2)\Lambda_0 + 2\Lambda_1, (\ell-1)\Lambda_0 + \Lambda_1, \\ &\quad \ell\Lambda_0, (\ell-1)\Lambda_0 + \Lambda_1, \ell\Lambda_0) \\ \eta(p^{(1)}) &= (\hat{0}, \hat{0}, \hat{1}, \hat{1}, \hat{0}, \hat{1}) \end{aligned}$$

We can represent the paths of the above examples as follows:



### 2.1.1 Weighted paths

Let  $p$  be a path, and  $\bar{p}$  the corresponding ground-state path in  $\mathbf{P}_\ell^L$ , with integer sequences  $\eta(p) = (\eta_0, \dots, \eta_L)$ , and  $\eta(\bar{p}) = (\bar{\eta}_0, \dots, \bar{\eta}_L)$ , respectively. We can assign to each path a weight, called the *energy function* as follows. Introduce a functional  $H : p \rightarrow \mathbf{Z}$ :

$$H(p) = \sum_{j=1}^L j\theta(\eta_j - \eta_{j-1}), \quad (1)$$

where  $\theta$  is the step function

$$\theta(z) = \begin{cases} 0 & (z > 0) \\ 1 & (z \leq 0), \end{cases} \quad (2)$$

and define the energy  $E(p)$  of the path  $p$  as

$$E(p) = H(p) - H(\bar{p}). \quad (3)$$

Notice that, by construction, the ground state paths  $\bar{p}$  of the vacuum sectors, that we are interested in, have  $H(\bar{p}) = 0$ . All other paths in the same set  $\mathbf{P}_\ell^L$  have  $H(p) > 0$ . Comparing any path  $p \in \mathbf{P}_\ell^L$  to  $\bar{p}$ , one can see that local departures from  $\bar{p}$  ‘cost energy’, and that the further away from the origin the departure occurs, the higher is the cost.

### 2.1.2 Polynomial analogues of branching functions

We define polynomial analogues  $B_L(q)$ , of the branching functions or Virasoro characters for the vacuum modules of the cosets  $\widehat{sl(2)}_{\ell-1} \times \widehat{sl(2)}_1 / \widehat{sl(2)}_\ell$  as the generating function of the weighted paths in  $\mathbf{P}_\ell^L$  [19],

$$B_L(q) = \sum_{p \in \mathbf{P}_\ell^L} q^{E(p)}. \quad (4)$$

The paths that belong to the vacuum sectors, that we are interested in, begin and end at  $\ell\Lambda_0$ .

## 2.2 Paths as tableaux

A Young *diagram*  $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$  is a collection of cells arranged in rows of length  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \dots$ . A *standard* Young tableau is a Young diagram whose cells are occupied by integers that strictly increase along rows and along the columns. In this subsection, we present a bijection between the paths and standard tableaux, which allows us to use the latter to represent the former.

To each path  $p$ , we associate a standard tableau  $T_p$  as follows: scan the sequence of integers  $\eta(p)$  that label a path  $p$ , and place the integer  $j$  in the  $(i+1)^{th}$  row, if and only if  $\eta_j = \hat{i}$ .

We can adhere to normal standard tableaux in this paper (no skew shapes) since we always have the first integer in  $\eta(p) = \hat{0}$ , and the sequence of integers encoding such paths between dominant integral weights are *lattice permutations*.

**Definition 3 (lattice permutation)** *Consider an alphabet  $\alpha$  in  $n$  letters  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ . The letters are totally ordered in the sense that  $\alpha_i < \alpha_j$  if  $i < j$ . A lattice permutation is a word in  $\alpha$  such that, on scanning the letters that compose it from left to right,  $\alpha_i$  occurs at least as many times as  $\alpha_j$ , for  $i < j$ .*

Note that for a given  $p$ , the contribution of step  $j$ , to the path energy, is zero if the integer  $(j+1)$  lies *below*  $j$  in  $T_p$ . With this property in mind, we introduce two integer-valued functionals or *statistics* on the tableaux:  $\mathbf{c}(T)$  and  $\mathbf{p}(T)$ .

### 2.2.1 The charge of a tableaux, $\mathbf{c}(T)$ , and the Thomas functional $\mathbf{p}(T)$

The functional  $\mathbf{p}(T)$ , introduced by Thomas [20], is the sum of the positive integers  $i < n$  such that  $i+1$  lies in a column to the *right* of  $i$ . (See [21], p. 243.) Observe that for any entry  $j$  in a standard tableau  $T$ ,  $j+1$  lies *either* to the right of  $j$  *or* below it. Therefore,  $\mathbf{p}(T_p)$  picks up the positive contributions to  $\theta(\eta_j - \eta_{j-1})$  for all  $j$ .

The charge of a tableau,  $\mathbf{c}(T)$  was defined by Lascoux and Schützenberger [22] for a tableau  $T$  of shape  $\lambda$  and weight  $\mu$ . (The weight of a tableau is a sequence  $\{\mu_1, \mu_2, \dots, \mu_n\}$ , where  $\mu_i$  is the number of times  $i$  occurs in the tableau. In this paper we are interested in standard tableau, *i.e.* the case  $\mu = \{1, 1, \dots, 1\}$ .) We shall follow [21], page 242, where  $\mathbf{c}(T)$  is defined for any tableau whose weight  $\mu$  is a partition.

From a tableau  $T$ , read off the entries from right to left and from top to bottom to get a word  $w(T) = a_1 a_2 \dots a_n$ . Assign an *index* to each  $a_i$  as follows. Let 1 have index 0. If the number  $r$  has index  $i$ , then  $r+1$  will have index  $i$  if

it is to the *right* of  $r$ , and  $i + 1$  if it is to the *left*. The charge  $c(w)$  of a word  $w$  is the sum of its indices. The charge of a tableau  $T$  is the charge  $c(w(T))$ .

For example, for the tableau

$$T = \begin{array}{cccc} 1 & 2 & 4 & 8 \\ 3 & 6 & 9 & \\ 5 & 7 & & \end{array},$$

the word  $w(T) = 842196375$ . The indices of  $1, 2, \dots, 9$  are  $0, 1, 1, 2, 2, 3, 3, 4, 4$ , respectively and  $c(w(T)) = 20$ . Also,  $\mathbf{p}(T) = 1 + 3 + 5 + 7 = 16$ .

### 2.2.2 The Schützenberger involution

The involution we shall describe is a special case of a family of operations which are indexed by points  $p \in Z \times Z$  occupied by an ordered set of objects on which they act injectively. These were defined by Schützenberger and are called *jeu de taquin*. For the involution  $S$  on standard tableaux, also called *an evacuation*, the point in question is the top left corner  $(1, 1)$ , *e.g.*, the cell occupied by the number 1 in the above example.

The operation  $S$  is defined by the following set of moves. First, remove the entry in cell  $(1, 1)$ . Move the smaller of the two integers immediately below or to the right of it to occupy the cell  $(1, 1)$ . This leaves a vacancy in the cell from which the integer was moved. Then look for the smallest integer immediately below or to the right of this freshly vacated cell, and move that one in to squat in it. This leaves a new cell empty. In general, whenever the cell  $(i, j)$  is empty, and  $t(i_1, j_1) = \min(t(i + 1, j), t(i, j + 1))$ , slide the integer from  $(i_1, j_1)$  to  $(i, j)$ . And so on, until there are no more cells to the right and below the cell vacated. In this last cell insert the number  $n$  with parentheses around it, to distinguish it from the  $n$  that is already present in the tableau. Now, remove the (new) integer from the cell  $(1, 1)$ , and repeat this procedure, inserting  $(n - 1)$  in the last cell. And so on, until all the entries of the original tableau are deleted, and we have parentheses around the integers in the new standard tableau, which we then remove.

**Example 3 (An evacuation)** *In the tableau  $T$  above, remove the entry from the cell  $(1, 1)$  and start the evacuation:*

$$\begin{array}{cccc} \cdot & 2 & 4 & 8 \\ 3 & 6 & 9 & \\ 5 & 7 & & \end{array} \rightarrow \begin{array}{cccc} 2 & \cdot & 4 & 8 \\ 3 & 6 & 9 & \\ 5 & 7 & & \end{array} \rightarrow \begin{array}{cccc} 2 & 4 & \cdot & 8 \\ 3 & 6 & 9 & \\ 5 & 7 & & \end{array} \rightarrow \begin{array}{cccc} 2 & 4 & 8 & \bullet \\ 3 & 6 & 9 & \\ 5 & 7 & & \end{array}.$$

*Replace the  $\bullet$  by (9). Iterating this one more time, we get*

$$\begin{array}{cccc} \cdot & 4 & 8 & (9) \\ 3 & 6 & 9 & \\ 5 & 7 & & \end{array} \rightarrow \begin{array}{cccc} 3 & 4 & 8 & (9) \\ \cdot & 6 & 9 & \\ 5 & 7 & & \end{array} \rightarrow \begin{array}{cccc} 3 & 4 & 8 & (9) \\ 5 & 6 & 9 & \\ \cdot & 7 & & \end{array} \rightarrow \begin{array}{cccc} 3 & 4 & 8 & (9) \\ 5 & 6 & 9 & \\ 7 & \bullet & & \end{array}.$$

After a couple of more steps, we have

$$\begin{array}{cccc} 3 & 4 & 8 & (9) \\ 5 & 6 & 9 & \\ 7 & (8) & & \end{array} \rightarrow \begin{array}{cccc} 4 & 6 & 8 & (9) \\ 5 & 9 & (7) & \\ 7 & (8) & & \end{array} \rightarrow \begin{array}{cccc} 5 & 6 & 8 & (9) \\ 7 & 9 & (7) & \\ (6) & (8) & & \end{array},$$

and so on until

$$T^S = \begin{array}{cccc} 1 & 3 & 5 & 9 \\ 2 & 4 & 7 & \\ & 6 & 8 & \end{array}.$$

It is possible to show that the following two properties in the above example:

i)  $S$  is an involution, *i.e.*  $S^2 = 1$ , or  $(T^S)^S = T$ .

ii)  $\mathbf{c}(T) = \mathbf{p}(T^S)$ .

Both of these properties are, in fact, true in general. See [23] and references therein.

### 2.2.3 Energy of a path as the statistic $\mathbf{p}$

The following is a key observation: Let a path  $p = (\eta_1, \dots, \eta_L)$  be encoded as a Young tableau  $T_p$ . The contributions to  $H(p)$  come from the integers  $j$  for which  $(j+1)$  is to the right of it in  $(T_p)$ . This, together with the definition of  $\mathbf{p}(T_p)$ , leads us to

**Proposition 1** *For a path  $p = (\eta_1, \dots, \eta_L)$  encoded as a Young tableau  $T_p$  with a total number of nodes  $|T_p| = L$ ,  $H(p) = \mathbf{p}(T_p)$ . The energy of the path  $E(p)$  is then*

$$E(p) = \mathbf{p}(T_p) - \frac{L}{2} \left( \frac{L}{2} - 1 \right).$$

*Proof:* Observe that for a path  $\bar{p} := (0, 1, 0, 1, \dots, 0, 1)$ , of length  $L$ , if the corresponding tableau is  $\bar{T}$ , then

$$\mathbf{p}(\bar{T}) = \sum_{i=1}^{L/2-1} (2i) = \frac{L}{2} \left( \frac{L}{2} - 1 \right).$$

The proof follows directly from the definition of  $E(p)$ . □

## 3 RTM objects: strings and tableaux

In [4] the eigenvalues of the row-to-row transfer matrix of the ABF models were obtained in terms of solutions to the Bethe equations. In order to solve the Bethe equations, one typically starts by *assuming* that the solutions form *strings* [24]: they form clusters in the complex plane, where each cluster of roots



has a common real part, and equally spaced imaginary parts. A cluster that consists of  $j$  of elements is called a string of length  $j$ .

It is standard [25] to multiply the Bethe equations for the components of each string, and end up with equations for the real parts of the solutions. The logarithm of the multiplied form of the Bethe equations is then taken such that the (half-) integer branches are *distinct*, hence referred to as *fermionic*.

Every eigenvector of the RTM can be described in terms of such a set of (half-) integers, which form convenient labels for describing the physics of these models. The characterization of roots in terms of the length of the strings and the (half-)integer branches of logarithms (also called Takahashi numbers) is called a *rigged configuration* in [9]. Let us express the indexing and counting of states in the language of [9],[10].

For a given two rowed partition (or Young diagram)  $\lambda = (\lambda_1, \lambda_2)$ ,  $\lambda_1 + \lambda_2 = L$ , we shall define another,  $\nu, \nu \vdash \lambda_2$  so that the rows of a partition  $\nu$  represent strings, and the lengths of these rows are viewed as the lengths of these strings. We shall require that the maximum length of strings allowed is  $\ell$ . Let  $m_j$  be the number of rows length  $j$ , and  $Q_j$  be the number of cells in the first  $j$  columns of diagram  $\nu$ .

The rows are labelled by integers  $J_{j,\mu}, \mu = 1, \dots, m_j, j = 1, \dots, \ell, 0 \leq J_{j,\mu} \leq P_j$ , and we require  $P_j \geq 0$  for a rigged configuration to be *admissible*. Unlike the fermionic Takahashi numbers, these integers  $J_{j,\mu}$ , however, are allowed to repeat, *i.e.* they may be called *bosonic*.<sup>8</sup>

The maximum allowed *bosonic* integer  $P_j$  for any of the labels of the  $m_j$  strings of length  $j$  is given by  $P_j = L - 2Q_j$ . Note,  $Q_j = \sum_{i=1}^{\ell} \min(i, j) m_i$  and therefore,

$$P_j = L - 2 \sum_{i=1}^j \sum_{k=i}^{\ell} m_k \quad (5)$$

$$= (\lambda_1 - \lambda_2) + 2 \sum_{k>j}^{\ell} (k - j) m_k. \quad (6)$$

**Definition 4 (rigged configurations)** *A configuration of partitions  $\nu$  whose rows are indexed by non-negative integers  $J_{j,\mu}, \mu = 1, \dots, m_j$  no larger than  $P_j$  for  $1 \leq j \leq \ell$  where  $P_j$  is calculated as above is called a rigged configuration. We shall denote a rigged configuration by  $(\nu, L, \{J\})$*

The total number of such rigged configurations is given by

$$\sum_{m_1, \dots, m_{\ell}} \prod_{j=1}^{\ell} \binom{P_j + m_j}{m_j} \quad (7)$$

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<sup>8</sup>For a fixed number of particles, the labels bosonic or fermionic do not, however, mean very much. It is only when the particle number changes that the new occupation numbers for bosons or fermions or any other 'ons' exhibit their statistical character.

### Remark

The counting procedure above is identical to the one involving Takahashi integers  $I_{j,\mu}$  (see[26]), which are related to the (bosonic) integers  $J_{j,\mu}$  by

$$I_{j,\mu} = J_{j,\mu} + \mu - \frac{1}{2}(P_j + m_j + 1).$$

### 3.1 The quasi-particle momentum of a rigged configuration

While the sum of the integers gives the momentum of the eigenstate, fermionic character sums are generating functions for the momenta of sectors of the excited states in the theory, chosen in an appropriate way. These excited states are labelled by their quasi-particle content, and for the correct counting of quasi-particle momenta, it is necessary to assign the appropriate zero-momentum point. This is done by subtracting off the *Fermi momentum*, and for our models, and the excitations counted by the variable  $P_k$ , the edge is given by the  $L$  dependent piece in equation . We therefore have to take the following sum in order to get the momentum of the excited states counted by  $P_k$ :

$$P(\nu, \{I\}) = \sum_{j=1}^{\ell-1} \sum_{\alpha=1}^{P_j} (I_{j,\alpha} - \frac{L}{4} \delta_{1j}).$$

A useful way of viewing this sum for the momentum of the quasiparticle states is to write it in terms of the variables  $J_{j,\alpha}$  decompose it into a piece which has all the  $J_{j,\alpha} = 0$  and a sum over the integers  $J_{j,\alpha}$ . That is,

$$P(\nu, \{J\}) = \frac{1}{4} \sum_{i,j=1}^{\ell-1} P_i C_{i,j} P_j + \sum_{\alpha=1}^{P_j} J_{j,\alpha}, \quad (8)$$

where  $C_{i,j}$  are the elements of the Cartan matrix for the  $A_{l-1}$  root system.

### 3.2 Rigged configurations and lattice permutations

Before launching into the description of the bijective correspondence between rigged configurations and tableaux, or equivalently, lattice permutations, let us first prove the following proposition.

**Proposition 2** *A rigged configuration is admissible if and only if the tableau associated to it is a standard tableau.*

*Proof:* Let us assume that the tableau  $T$  associated to a sequence of 0's and 1's is not standard, *i.e.*,  $\lambda_1 - \lambda_2 < 0$  for the shape  $\lambda = (\lambda_1, \lambda_2)$  of  $T$ . This implies

$P_\ell < 0$ . Conversely, note that

$$P_{j_1} - P_{j_2} = 2 \sum_{k=j_1+1}^{j_2} (k - j_1)m_k + 2 \sum_{k=j_2+1}^{\ell} (j_2 - j_1)m_k \geq 0, \quad j_1 < j_2.$$

This implies that if the configuration is not admissible,  $P_j < 0$  for some  $j \Rightarrow P_\ell < 0$  and therefore  $\lambda_1 - \lambda_2 < 0$ .  $\square$

### 3.2.1 From a lattice permutation to a rigged configuration

A key idea, used repeatedly is that of a *special string* or row of a rigged configuration.

**Definition 5 (special string)** *A row in  $\nu$  of length  $i$  and label  $\alpha, \alpha = 1, \dots, m_i$  in a rigged configuration  $(\nu, L, \{J\})$  is called ‘special’ if its rigging integer is maximal, i.e.  $J_{i,\alpha} = P_i$ .*

From the sequence of integers  $w = b_1 b_2 \dots b_L$ , we shall construct the rigged configuration that corresponds to it by an inductive process due to Kerov, Kirillov and Reshetikhin [9].

Let us assume that we have already constructed  $(\nu, k-1, \{J\})$  from  $w_1 = b_1 b_2 \dots b_{k-1}$ . Order the rows of equal length so that the integers within each such block are weakly decreasing from the top. So, of the rows are numbered  $1, \dots, m_j$  from the top of a block of  $j$ -strings,

$$P_j \geq J_{j,1} \geq \dots \geq J_{j,m_j} \geq 0 \quad \forall j. \quad (9)$$

We shall now describe how to get to  $(\tilde{\nu}, k, \{\tilde{J}\})$ , given the value of  $b_k$ . There are two possibilities:  $b_k = 0, 1$ . For  $b_k = 0$ , set  $\tilde{\nu} = \nu$  and every rigging integer  $\tilde{J} = J$  (we suppress the labels here). The only change is therefore,  $k-1 \mapsto k$ , and therefore  $P_j \mapsto P_j + 1, \forall j$ .

If  $b_k = 1$ , set  $i^* = \max\{i | J_{i,1} = P_i\}$  and  $r^* = 1 + \sum_{i > i^*} m_i$ . Then

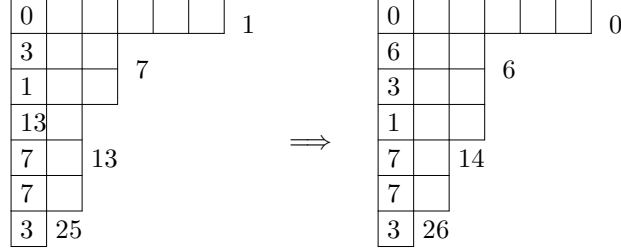
$$\nu = (\nu_1, \dots, \nu_{Q_1}) \mapsto \tilde{\nu} = (\nu_1, \dots, \nu_{r^*} + 1, \dots, \nu_{Q_1}).$$

In words, we add a box to the highest special row.

Next, we need to describe  $\{\tilde{J}\}$ . We set  $\tilde{J} = J$  for all rows of  $\tilde{\nu}$  except the  $r^{*th}$  row, which we set equal to the maximum allowed. Note that the following changes occur to the maximum integers  $P_j$ :

$$P_j \mapsto \tilde{P}_j = \begin{cases} P_j + 1, & (j \leq i^*) \\ P_j - 1, & (j > i^*). \end{cases} \quad (10)$$

Thus the new (changed) integer is set equal to  $\tilde{P}_{i^*+1} = P_{i^*+1} - 1$ . Once again, we re-order the rows of length  $i^* + 1$  so that the integers satisfy (9). From (10) it is also clear that  $\tilde{P}_j \geq 0 \forall j$ , so that the new configuration is also admissible.



**Example 4** ( $b_k = 1$ )  $(\nu, L = 39) \Rightarrow (\nu', 40)$ , for  $b_{40} = 1$ . A node is added to the highest special row, and the resulting configuration has its rows rearranged so that the integer riggings are weakly decreasing.

### 3.2.2 From a rigged configuration to a lattice permutation

Here we describe a way of reading off  $w_k$  from a rigged configuration  $(\nu, k, \{J\})$ , where  $w_k$  is either a 0 or a 1, and modifying it to get another configuration  $(\bar{\nu}, k-1, \{\bar{J}\})$ , a process called *ramification*. We can perform the same procedure on  $(\bar{\nu}, k-1, \{\bar{J}\})$ , to extract  $w_{k-1}$ , so this is a recursive method for generating a sequence of 0's and 1's from a given rigged configuration  $(\nu, L, \{J\})$ .

Here we describe the *ramification rules*. First, we arrange the rows of equal length  $j$  so that the integers are in a weakly increasing order from the top:

$$0 \geq J_{j,1} \leq \dots \leq J_{j,m_j} \leq P_j \quad \forall j. \quad (11)$$

Set  $i_* = \min\{i | J_{i,m_i} = P_i\}$  and  $r_* = \sum_{i \geq i_*} m_i$ , and also assign

$$\nu = (\nu_1, \dots, \nu_{Q_1}) \mapsto \bar{\nu} = (\nu_1, \dots, \nu_{r_*} - 1, \dots, \nu_{Q_1}).$$

In words, we delete a box from the lowest special row.

Next, we need to describe  $\{\bar{J}\}$ . We set  $\bar{J} = J$  for all rows of  $\bar{\nu}$  except the  $r_*^{th}$  row, which we set equal to the maximum allowed. Note that the following changes occur to the maximum integers  $P_j$ :

$$P_j \mapsto \bar{P}_j = \begin{cases} P_j - 1, & (j < i^*) \\ P_j + 1, & (j \geq i^*). \end{cases} \quad (12)$$

Thus the new (changed) integer is set equal to  $\tilde{P}_{i^*-1} = P_{i^*-1} - 1$ . Once again, we re-order the rows of length  $i^* + 1$  so that the integers satisfy (11). From (12) it is also clear that  $\bar{P}_j \geq 0 \forall j$ , so that the new configuration is also admissible.

We can now define  $w_k := 0$  if  $i_* = 0$ , *i.e.*, if there are no special strings in  $\nu$ , or  $w_k := 1$  if there is at least one such special string ( $i_* > 0$ ).

## 4 The bijection between paths and rigged configurations

The starting point of our proposed bijection between the CTM paths, and the RTM rigged configurations, are the following observations: Once we know that the Bethe Ansatz solution of a model can be cast in terms of interacting quasi-particles [5], one can attempt re-phrase the CTM solution of the same model in that same language. The ABF model labelled by  $\ell$  contains strings of length  $1, \dots, \ell$ . Looking at the CTM paths of such a model, one can think of the *peaks* of different heights as representing the various types of strings. (One can think of the peaks of height 1 as *non-physical*: they are there to fill the *Dirac* (or *Fermi sea*, but do not contribute to the total momentum of the physical states.)

In this picture, each path encodes the *momentum* of a configuration of strings. It is now clear that one can classify the set of all paths into sectors, such that, the set of all paths in each sector share the same number of string-lengths, and therefore quasi-particles of each type, but the different paths in each sector correspond to different excited states of the same set of quasi-particles.

It turns out, and it is easy to see by inspection, that each sector will contain a certain unique path, with lowest possible total momentum. We wish to refer to this as a *minimal path configuration*. Furthermore, since we are working on finite paths, each sector will also contain a unique path, with largest possible total momentum. We wish to refer to that as a *maximal configuration*. As we will see below, identical remarks can be made about the rigged configurations, and for a rigged configuration of a certain set of strings, one can define a minimal and a maximal configuration. In order to identify the paths and the corresponding rigged configurations, we need to make sure that we do have a weight-preserving bijection.

We proceed in two steps: firstly, we show that, sector by sector, the minimal and maximal path and rigged configurations are in bijection. Secondly, we show that, sector by sector, the rest of the configurations are also in bijection with the other paths.

### 4.1 The minimal configurations

Let  $(\nu, L, \{J = 0\})$  be a rigged configuration with a string content  $(m_1, m_2, \dots, m_\ell)$ , with all the rigging integers set to zero. Since some of the  $m_i$  could be zero, we shall label the rows from  $i = 1$  (the first row) to  $i = Q_1 = m_\ell + m_{\ell-1} + \dots + m_1$  (the bottom row) and let  $\nu_i$  be the length of the  $i^{th}$  row. We thus have  $\nu_1 \geq \nu_2 \geq \dots \geq \nu_{Q_1}$ . Once again, we restrict ourselves to the vacuum module, which is characterized by  $\lambda_1 - \lambda_2 = 0$  for even  $L$ .

Let us call a *minimal* rigged configuration  $(\nu, L, \{J = 0\})$  with all the integers set to zero. This is the state with the smallest value of the momentum ascribed to the *excitations*. Let us evaluate what the sequence of ranks is under the

ramification rules that map this rigged configuration into a standard tableau. Order the rows according to (11).

Note that  $P_{\nu_1} = 0$  for  $L$  even and  $P_j > 0$  for  $j < \nu_1$ . Therefore  $w_L = 1$ , and we delete the rightmost cell from the row  $\nu_{m_{\nu_1}}$ . The modified rigged configuration now has a row of length  $\nu_1 - 1$  which is special by construction, while  $P_{\nu_1} \mapsto P_{\nu_1} + 1$ . Therefore,  $w_j = 1, j = L, L-1, \dots, L - \nu_1 + 1$ , until the top row has been *completely removed*, by which stage,  $P_{\nu_2} = \nu_1$ . Before  $P_{\nu_2}$  is brought back to 0,  $w_j = 0, j = L - \nu_1, L - \nu_1 - 1, \dots, L - 2\nu_1 + 1$ . Thus, from a minimal configuration, strings of length  $k$  generate  $k$  0's and  $k$  1's in succession, and the longer strings are deleted first.

In this way we end up with a word  $w$ :

$$w = w_L w_{L-1} \cdots w_1 = 11 \cdots 100 \cdots 0 \underbrace{11 \cdots 1}_{\nu_i} \underbrace{00 \cdots 0}_{\nu_i} 1 \cdots 101 \cdots 10,$$

where the first sequence of 1's and the first sequence of 0's are each of length  $\nu_1$ , the next sequence of 0's and 1's of length  $\nu_2$  each, and so on. In general, for a minimal configuration as defined, there are sequences of  $\nu_i$  0's followed by  $\nu_i$  1's for  $\nu \vdash \frac{L}{2}$ .

### Remark

Given an admissible rigged configuration  $(\nu, L, \{J\})$ , we define another,  $(\nu, L, \{J^\sigma\})$  by performing the following operations (involutions)  $\sigma_{j,\alpha}$  ( $j \in \{1, \dots, \ell\}$ ,  $\alpha \in \{1, \dots, m_j\}$ ) and  $\sigma := \prod_{j=1}^\ell \prod_{\alpha=1}^{m_j} \sigma_{j,\alpha}$ :

$$\begin{aligned} \sigma_{j,\mu} : \{J\} &\longrightarrow \{J\} \\ J_{j,\mu} &\mapsto J_{j,\mu}^\sigma = P_j - J_{j,m_j+1-\mu}. \end{aligned}$$

Note that  $\sigma_{j,\alpha}^2 = \text{id}$ .

Recall that a node is added to a row in  $\nu$  if it is the highest special row, and this results in a decrease in the maximum integer allowed for all rows above it. Every uninterrupted sequence of say  $x$  1's would require the addition of cells to the *same* row, since by construction it would be assigned an integer equal to its maximum allowed, provided the integers rigging the rows above it are less than the maximum allowed integer by at least  $x$ . Thus every sequence of  $(\nu_i)$  1's would make a  $\nu_i$ -string pop up, but in order for the word of 0's and 1's to be a lattice permutation, every such sequence must be preceded by an equal number of 0's. Note also, that every row in the configuration generated must be special by construction.

It is clear from the above discussion that  $\sigma_{i,m_i}(\nu, L, \{J = 0\})$  produces a word which can be obtained from the minimal configuration in the following way. The first occurring sequence of  $\nu_i$  1's and  $\nu_i$  0's (*i.e.*, that which starts at  $w_k$  for the largest  $k$ ) is translated so as to start from  $w_L$ .

## 4.2 The maximal configurations:

What about the maximal configuration, the one in which all the entries  $\{J_{j,\alpha} = P_j\} \forall j$ ? The previous remark makes it clear that if  $\sigma$  which is the product of *all* the involutions  $\sigma_{j,\alpha}$  is applied, we would reach the maximal configuration, which is a lattice permutation of the form

$$w = w_L w_{L-1} \cdots w_1 = 1010101 \cdots 01100 \cdots \underbrace{11 \cdots 1}_{\nu_i} \underbrace{00 \cdots 0}_{\nu_i} 11 \cdots 100 \cdots 0,$$

Thus, we conclude that the lattice permutations corresponding to  $\{J = 0\}$  and  $\{J = P\}$ ,  $w_{min} = w_1 w_2 \cdots w_L$  and  $w_{max} = w'_1 w'_2 \cdots w'_L$  respectively, are related by the evacuation involution  $w'_i = 1 - w_{L+1-i}$ .

## 4.3 The rest of the configurations

Now we have to deal with the rest of the configurations – we wish to show that any path, that is neither minimal nor maximal, maps to a rigged configuration, that is also neither minimal nor maximal, and *vice versa*.

More importantly, we want to show that if the ABF path has maximal height  $\ell + 1$ , then the maximal string length is necessarily  $\ell + 1$ . Let the length of the longest string be  $l_{max}$ . If at some stage, the height reached is  $\ell + 1$ , then the corresponding partial tableau is such that  $\lambda_1 - \lambda_2 = \ell + 1$ , and therefore  $P_{l_{max}} = \ell + 1$  from equation 5.

We know that for the rectangular tableaux (equal number of 0 and 1 occurrences) which defines the vacuum sector,  $P_{l_{max}} = \lambda_1 - \lambda_2 = 0$ . Also, for every deletion from any row in  $(\nu, \{J\})$ , the corresponding  $P_{l_{max}}$  goes *up* by 1, and if the length  $L$  is reduced by one, leaving the shape of  $\nu$  unchanged, all the  $P_j$ s would go *down* by 1. Thus, to go from 0 to  $\ell + 1$ , in order to reach the configuration described above, the number of deletions from the rows of  $\nu$  must be  $\ell + 1$  more than reductions in  $L$ . Every time a row smaller than the longest is special, and cells have to be deleted from it, the special string disappears quicker than the longest one would have (obviously). Therefore, the next step just after this disappearance would have to be a 0, and the length is reduced without deleting cells from  $\nu$ . Therefore, the only way  $P_{l_{max}}$  can go from 0 to  $\ell + 1$  is by deleting from a row of length at least  $\ell + 1$ . However, if the row has more than  $\ell + 1$  cells, then the maximum height in the path picture would then be more than  $\ell + 1$ .

## 4.4 Weighted sums

In [10], the following identity was established, based on the bijection described above:

$$\sum_{T \in SYT(\lambda)} q^{\mathbf{p}(T)} = \sum_{T \in SYT(\lambda)} q^{\mathbf{c}(T)} = \sum_{\nu} \prod_{j=1}^{\ell} q^{c(\nu, L)} \begin{bmatrix} s_j + P_j \\ s_j \end{bmatrix}. \quad (13)$$

where

$$c(\nu, L) = \frac{1}{2}L(L-1) - \frac{1}{2}LQ_1 - \sum_i m_i P_i, \quad (14)$$

and

$$\begin{bmatrix} A+B \\ B \end{bmatrix} = \frac{\prod_{j=1}^{A+B} (1-q^j)}{\prod_{j=1}^A (1-q^j) \prod_{j=1}^B (1-q^j)}.$$

By the equality of the energy functional on paths,  $H$  with  $\mathbf{p} - \frac{L}{2}(\frac{L}{2} - 1)$ , we arrive at a generating functional for counting CTM paths as follows. Observe that  $Q_j - Q_{j-1} = \sum_{k \geq j} m_k = -\frac{1}{2}(P_{j-1} - P_j)$ , so that

$$m_j = \frac{L}{2}\delta_{j1} - \frac{1}{2} \sum_k C_{jk} P_k$$

and thus,

$$c(\nu, L) = \frac{1}{2}L(L-1) - \frac{1}{4}L^2 + \frac{1}{4} \sum_{ij} P_i C_{ij} P_j. \quad (15)$$

This establishes the equality of the energy of the CTM path and the quasiparticle momentum of a rigged configuration (8).

In order to obtain the Virasoro character it is necessary to subtract off the Fermi momentum, or in the path language the energy of the ground state. For the ground state,  $P_j = 0 \forall j$  and  $m_j = (L/2)\delta_{j1}$ , so that  $Q_1 = (L/2)$ . After subtraction, we arrive at the proof of

$$B_L(\ell\Lambda_0) = \sum_{\nu} \prod_{j=1}^{\ell} q^{\frac{1}{4} \sum_{ij} P_i C_{ij} P_j} \left[ \frac{1}{2}L\delta_{j1} - \frac{1}{2} \sum_k C_{jk} P_k \right]. \quad (16)$$

### Remark

The observation that Thomas' statistic  $\mathbf{p}$  gives the required energy function  $H$  of a path ensures that the bijection of [9, 10] will necessarily generalize the proof of the bosonic-fermionic identity for the higher rank cases. (See [27].)

## 5 Discussion

The point of this work was to establish an explicit weight-preserving bijection between two sets of combinatorial objects: The paths that parameterize the CTM eigenvectors, and the strings, or rigged configurations, that parametrize the RTM eigenvectors. In this bijection, standard tableaux play the crucial role of interpolating objects. The central point of this work, is that weights associated with each of the above objects: the energies of the paths, the momenta of the strings, and the Thomas statistic on the standard tableaux (which



are related by an involution to the Lascoux-Schützenberger charge), are equal. Though our work is restricted to the vacuum sectors of the regime-III ABF models, this bijection is so natural, that we conjecture that analogous bijections exist for the rest of the sectors, and for more general models.

### Note added

After this work was completed, we received a preprint [28], where the relation between the energies of the CTM paths, and the Lascoux-Schützenberger charge of tableaux was noted, and used for somewhat different purposes.

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